Introduction to Formal Methods

Lecture 5
From Programs to Formulas

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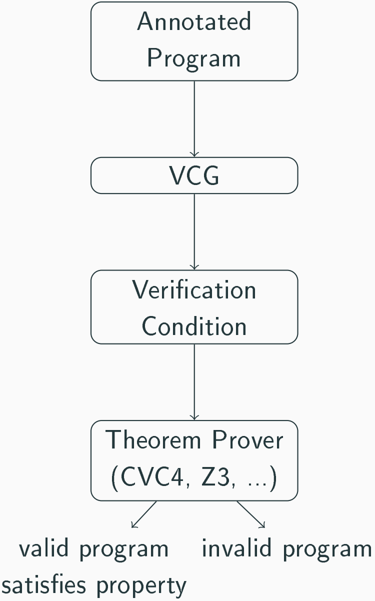
Verification-Condition Generation



Steps in Verification

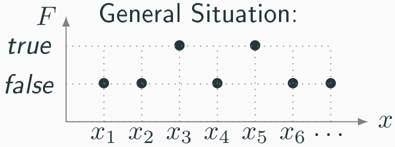
- Generate formula whose validity implies correctness of program
- Attempt to prove formula
  - If formula is **valid**, program is correct
  - If formula has a **counterexample**, it indicates one of these:
    - error in the program
    - error in the property
    - error in auxiliary statements (e.g. loop invariants)

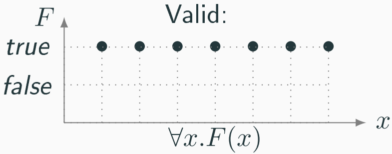
Terminology

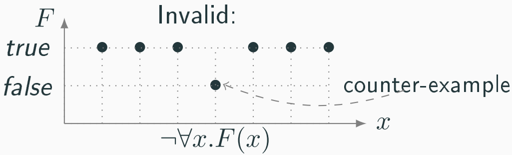
- Generated formulas:
  verification conditions
- Generation process:
  verification-condition generation
- Program that generates formulas:
  Verification-Condition Generator(VCG)

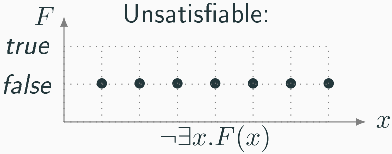
Validity and Satisfiability

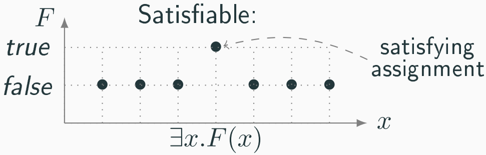
$F(x)$: formula with free variable(x) x











F is valid $\Leftrightarrow$ $\overline{F}$ is unsatisfiable

F is invalid $\Leftrightarrow$ $\overline{F}$ is satisfiable

Verification Condition Generation Example

We examine algorithms for going from programs to their verification conditions

```
if (x > 0)
        res = x * 2 + 1;
else
        res = 24;
assert(res > 0);
```

For the following formula, we check validity:
all variables are universally quantified

$$\Big(((x > 0) \land (\mathit{res} = 2x + 1)) \lor (\neg(x > 0) \land (\mathit{res} = 24))\Big) \to (\mathit{res} > 0)$$

Simple Programming Language

```
x = T
if (F) c1 else c2
c1 ; c2
while (F) c1
```

ordinary control structure

$$
\begin{aligned}
c \quad &::= x = T \mid (\textbf{if } (F)\ c \textbf{ else } c) \mid c; c \mid (\textbf{while } (F)\ c) \\
T \quad &::= K \mid V \mid (T + T) \mid (T - T) \mid (K * T) \mid (T/K) \mid (T\%K) \\
F \quad &::= (T == T) \mid (T < T) \mid (T > T) \mid (\ F) \mid (F \&\& F) \mid (F || F) \\
V \quad &::= x \mid y \mid z \mid \cdots \\
K \quad &::= 0 \mid 1 \mid 2 \mid \cdots
\end{aligned}
$$

terms like in Integer arithmetic

Boolean formulas without quantifiers

Collatz Conjecture

Prove this program always terminates for any natural number x:

```
while (x > 1) {
  if (x % 2 == 0) x = x / 2;
    else x = 3*x+1;
}
```

"Mathematics is not yet ripe for such problems" -Paul Erdös

Remark: Turing-Completeness

- This language is Turing-complete
- Every possible program (Turing machine) can be encoded into computation on integers (computed integers can become very large)
- Problem of taking a program and checking whether it terminates is undecidable
- **Rice**'s Theorem: all properties of programs that are expressed in terms of the results that the programs compute (and not in terms of the structure of programs) are undecidable

Remark: Turing-Completeness

In real programming languages we have bounded integers, but we have other sources of unboundedness, e.g.

- `BigInt` data type of Java and Scala (sequence of digits of any length)
- example: sizes of linked lists and of other data structure
- Program syntax trees for an interpreter or compiler (we would like to handle programs of any size)

What is Decidable

- Checking satisfiability of Presburger arithmetic formulas (even with quantifiers) is decidable
- Checking if there exists an input to a program in our language for which program computes a given value (e.g. 1) is undecidable
- Quantifiers in Presburger arithmetic cannot be used to define `z=x*y`
  - But we can write a program that computes `x*z` and stores it in `z`
- Programs without loops can be translated into Presburger arithmetic
- Loops give much more expressive power to Presburger arithmetic than quantifiers
  (situation can be different if we did not work with Presburger arithmetic)

VC Generation for Programs

- Program can be represented by a formula relating initial and final state
- Consider program with variables `x`, `y`, `z`

program:	`x = x + 2; y = x + 12`
relation:	$\{(x, y, z, x', y', z') \mid x' = x + 2 \land y' = x' + 12 \land z' = z\}$
formula:	$x' = x + 2 \land y' = x' + 12 \land z' = z$

Examples

Relation between initial and all possible final states

```
x = x + 3;
x = x + 2;
```

$$\{(x, x') \mid x' = x + 5\}$$

```
x = x + x;
```

$$\{(x, x') \mid x' = 2x\}$$

```
while (x != 10) {
  x = x + 1;
}
```

$$\{(x, x') \mid x \leq 10 \wedge x' = 10\}$$

```
while (5 == 5) {
  x = x;
}
```

$$\emptyset$$

Why Relations

The meaning is, in general, an arbitrary relation. Therefore:

- For certain states there will be no results
  - In particular, if a computation starting at a state does not terminate
- For certain states there will be multiple results
  - This means execution starting in a state will sometimes compute one and sometimes other result
  - Verification of such program must account for both possibilities
- Multiple results are important for modeling e.g. concurrency, as well as approximating behavior that we do not know
  (e.g. what the operating system or environment will do, or what the result of complex computation is)

Example of Non-Determinism

```
x = randomInteger()
if (x > 10) {
 y = y+1
} else {
 y = y+2
}
```

- Relation between the initial and the final y:

$$\{(y, y') \mid (y' = y + 1 \vee y' = y + 2)\} =$$

$$\{\cdots, (100, 101), (100, 102), (101, 102), (101, 103), \cdots\}$$

- obviously, not a function

Relations

- Cartesian product: $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$
- Relation $r \subseteq A \times B$
- Diagonal relation: $\Delta_A = \{(x, x) | x \in A\}$
- Partial function $f : A \hookrightarrow B$

$$\forall x \in A, y_1 \in B, y_2 \in B.(x, y_1) \in f \wedge (x, y_2) \in f \to y_1 = y_2$$

- Partial function is total iff

$$\forall x \in A. \exists y \in B.(x, y) \in r$$

- Function $f : A \to B$ when f is partial function and total on $A \times B$

Function Updates

$\mathsf{dom}(r) =$	$\{x \mid \exists y.(x, y) \in r\}$	domain
$\mathsf{ran}(r) =$	$\{y \mid \exists x.(x, y) \in r\}$	range

- $f : A \hookrightarrow B$, $g :\hookrightarrow B$

$$f \oplus g = \Big\{(x, y) \mid \big((x, y) \in f \land x \notin \mathsf{dom}(g)\big) \lor (x, y) \in g\Big\}$$

- $f[x := v]$ means $f \oplus \{(x, v)\}$

$$(f[x := v])(y) = \begin{cases} v & \text{if } \; y = x \\ f(y) & \text{if } \; y \neq x \end{cases}$$

A Simple Property

$$\begin{array}{llcl} \text{Relation Composition:} & t \circ r & = & \{(x, z) \mid \exists y.(x, y) \in t \wedge (y, z) \in r\} \\ \text{Relation Image:} & S \bullet r & = & \{y \mid \exists x \in S.(x, y) \in r\} \end{array}$$

Theorem. For $r \subseteq A \times A$ and $S \subseteq A$

$$S \bullet r = \mathsf{ran}(\Delta_S \circ r)$$

Transitive Closure

$$
\begin{aligned}
r &\subseteq A^2 \\
r^0 &= \Delta_A \\
r^1 &= r \circ \Delta_A = r \\
r^{n+1} &= r \circ r^n = r^n \circ r \\
r* &= \bigcup_{i \geq 0} r^i = \Delta_A \cup r \cup r^2 \cup \cdots
\end{aligned}
$$

Theorem.

$$\bigcap\{S \mid \Delta_A \cup S \circ r \subseteq S\} = r*$$

($r*$ is the least S satisfying the recursive condition)

Guarded Command Language

`assume(F)`	block all executions where `F` does not hold
`s1 ; s2`	do first `s1`, then `s2`
`s1 [] s2`	do either `s1` or `s2` arbitrarily
`s*`	execute `s` zero, once, or more times

Formula that Describe Relations

- c imperative command
- $R(c)$ formula describing relation between initial and final states of execution of c
- If $\rho(c)$ describes the relation, then $R(c)$ is formula such that

$$\rho(c) = \{(\vec{v}, \vec{v'}) \mid R(c)\}$$

- $R(c)$ is a formula between unprimed variables $\vec{v}$ and primed variables $\vec{v'}$

Formula for Assignment

$$x = t$$

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$$x = t$$

$R(x = t)$:

$$x' = t \land \bigwedge_{v \in V \setminus \{x\}} v = v'$$

- Note that the formula must explicitly state which variables remain the same (here: all except `x`)
- Otherwise, those variables would not be constrained by the relation, so they could take arbitrary value in the state after the command

assume

$$\texttt{assume}(F)$$

assume

$$\texttt{assume}(F)$$

$R(\texttt{assume}(F))$:

$$F \wedge \bigwedge_{v \in V} v = v'$$

$\rho(\texttt{assume}(F))$:

$$\Delta_{S(F)}$$

where $S(F) = \{\vec{v} \mid F\}$

Non-deterministic Choice

$$c_1 \mathbin{[\!]} c_2$$

Non-deterministic Choice

$$c_1 \mathbin{[\,]} c_2$$

$R(c_1 \mathbin{[\,]} c_2)$:

$$R(c_1) \vee R(c_2)$$

$\rho(c_1 \mathbin{[\,]} c_2)$:

$$\rho(c_1) \cup \rho(c_2)$$

- translation is simply a disjunction - this is why construct is interesting
- corresponds to branching in control-flow graphs

Sequential Commands

$$c_1; c_2$$

Reminder about relation composition and its definition:

$$r_1 \circ r_2 = \{(a, c) \mid \exists b.(a, b) \in r_1 \land (b, c) \in r_2\}$$

What are $R(c_1)$ and $R(c_2)$ and in terms of which variables they are expressed?

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What are $R(c_1)$ and $R(c_2)$ and in terms of which variables they are expressed?

$R(c_1; c_2)$:

$$\exists \vec{z}. R(c_1)[\vec{x'} := \vec{z}] \land R(c_2)[\vec{x} := \vec{z}]$$

where $\vec{z}$ are freshly picked names of intermediate states

if condition

```
if (F)
  s1
    else
      s2
```

$$(\mathtt{assume}(F); s1)$$

$$\square$$

$$(\mathtt{assume}(\neg F); s2)$$

$$\begin{tikzcd}
& {} \arrow[dl, "{[f]}"'] \arrow[dr, "{[\neg f]}"] & \\
{} \arrow[dr, "s_1"'] & & {} \arrow[dl, "s_2"] \\
& {} &
\end{tikzcd}$$

Example: Absolute Value

```
if (x > 0)
  y = x;
else
  y = -x;
```

One-Point Rule

- Assignments and assume statements generate equalities, many of which can be eliminated by one-point rule

$$(\exists x. x = t \wedge F) \leftrightarrow F[x := t]$$

- There are more complex quantifier elimination procedures that can be used in principle as well

Towards meaning of loops: unfolding

Loops can describe an infinite number of basic paths

Consider loop

$$L \equiv \textbf{while}(F)c$$

We would like to have

$$\begin{aligned} L &\equiv \textbf{if}(F)(c; L) \\ &\equiv \textbf{if}(F)(c; \textbf{if}(F)(c; L)) \end{aligned}$$

For $r_L = \rho(L)$,$r_c = \rho(c)$, $\Delta_f = \Delta_{S(F)}$, $\Delta_{\mathit{nf}} = \Delta_{S(\neg F)}$ we have

$$\begin{aligned} r_L &= (\Delta_f \circ r_c \circ r_L) \cup \Delta_{\mathit{nf}} \\ &= (\Delta_f \circ r_c \circ ((\Delta_f \circ r_c \circ r_L) \cup \Delta_{\mathit{nf}})) \cup \Delta_{\mathit{nf}} \\ &= \Delta_{\mathit{nf}} \cup \\ &\quad (\Delta_f \circ r_c) \circ \Delta_{\mathit{nf}} \cup \\ &\quad (\Delta_f \circ r_c)^2 \circ r_L \end{aligned}$$

Unfolding Loops

$$\begin{array}{rcl} r_L & = & \Delta_{\textit{\textbf{nf}}} \cup \\ & & (\Delta_f \circ r_c) \circ \Delta_{\textit{\textbf{nf}}} \cup \\ & & (\Delta_f \circ r_c)^2 \circ \Delta_{\textit{\textbf{nf}}} \cup \\ & & (\Delta_f \circ r_c)^3 \circ r_L \end{array}$$

We prove by induction that for every $n \geq 0$,

$$(\Delta_f \circ r_c)^n \circ \Delta_{\textit{\textbf{nf}}} \subseteq r_L$$

So, $(\Delta_f \circ r_c) * \circ \Delta_{\textit{\textbf{nf}}} \subseteq r_L$

We define r_L to be

$$r_L = (\Delta_f \circ r_c) * \circ \Delta_{\textit{\textbf{nf}}}$$

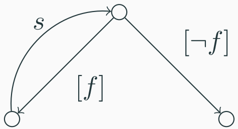
Therefore

$$\rho(\textbf{while}(F)c) = (\Delta_{S(F)} \circ \rho(c)) * \circ \Delta_{S(\neg F)}$$

While

```
while (F)
  s
```

$$(\mathtt{assume}(F); s)* \;; \\ (\mathtt{assume}(\neg F))$$



While

```
int x = 0;
while(x < 2)
        x = x + 1;
```

$$\left(\begin{array}{l} (x = 0) \vee \\ (x = 0 \wedge x < 2 \wedge x_1 = x + 1) \vee \\ (x = 0 \wedge x < 2 \wedge x_1 = x + 1 \wedge x_1 < 2 \wedge x_2 = x_1 + 1) \vee \\ (x = 0 \wedge x < 2 \wedge x_1 = x + 1 \wedge x_1 < 2 \wedge x_2 = x_1 + 1 \wedge x_2 < 2 \wedge x_3 = x_2 + 1) \vee \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \end{array}\right)$$

;

$x \geq 2$

While

```
int x = 0;
while(x < 2)
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```

$$\left(\begin{array}{l}
(x = 0) \vee \\
(x = 0 \wedge x < 2 \wedge x_1 = x + 1) \vee \\
(x = 0 \wedge x < 2 \wedge x_1 = x + 1 \wedge x_1 < 2 \wedge x_2 = x_1 + 1) \vee \\
(x = 0 \wedge x < 2 \wedge x_1 = x + 1 \wedge x_1 < 2 \wedge x_2 = x_1 + 1 \wedge x_2 < 2 \wedge x_3 = x_2 + 1) \vee \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{array}\right)$$

;

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While

```
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;

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While

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int x = 0;
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$$\left(\begin{array}{l}(x = 0 \wedge x \geq 2) \vee \\ (x = 0 \wedge x < 2 \wedge x_1 = x + 1 \wedge x_1 \geq 2) \vee \\ (x = 0 \wedge x < 2 \wedge x_1 = x + 1 \wedge x_1 < 2 \wedge x_2 = x_1 + 1 \wedge x_2 \geq 2) \\ \\ \\ \end{array}\right)$$

While

```
int x = 0;
while(x < 2)
        x = x + 1;
```

$$\left(\begin{array}{l} \cancel{(x = 0 \wedge x \geq 2) \vee} \\ \cancel{(x = 0 \wedge x < 2 \wedge x_1 = x + 1 \wedge x_1 \geq 2) \vee} \\ (x = 0 \wedge x < 2 \wedge x_1 = x + 1 \wedge x_1 < 2 \wedge x_2 = x_1 + 1 \wedge x_2 \geq 2) \\ \\ \\ \end{array}\right)$$

While

```
int x = 0;
while(x < 2)
        x = x + 1;
```

$$\left(\begin{array}{l} (x = 0 \land x \geq 2) \lor \\ (x = 0 \land x < 2 \land x_1 = x + 1 \land x_1 \geq 2) \lor \\ (x = 0 \land x < 2 \land x_1 = x + 1 \land x_1 < 2 \land x_2 = x_1 + 1 \land x_2 \geq 2) \\ \\ \\ \hspace{12em} x' = 2 \end{array}\right)$$

Havoc Statement

- Change a given variable arbitrarily

$$R(\text{havoc}(x)) = \{(x, y, z, x', y', z') \mid y' = y \wedge z' = z\}$$

- We can prove that the following equality holds when `x` does not occur in `E`
  `x = E` is `havoc(x); assume(x=E)`
- In other words, assigning a variable is the same as changing it arbitrarily and then assuming that it has the right value