Introduction to Formal Methods

Lecture 9

Verification by Solving Horn Clauses

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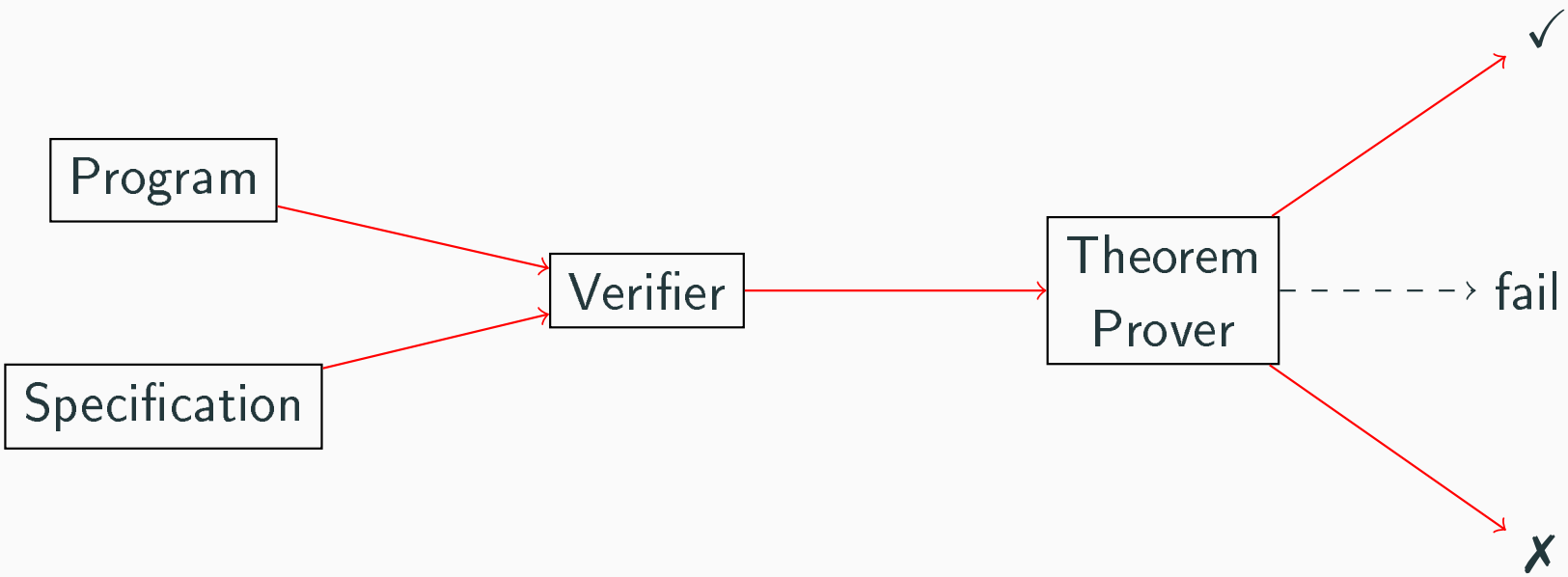
Recap: Automated Verification



Weakest Precondition Rules: Summary

c	$\mathsf{wp}(c, Q)$
$x := e$	$Q[x \mapsto e]$
$\mathsf{assume}(b)$	$b \to Q$
$\mathsf{assert}(b)$	$b \wedge Q$
$\mathsf{havoc}(x)$	$\forall y. Q[x \mapsto y]$
$c_1; c_2$	$\mathsf{wp}(c_1, \mathsf{wp}(c_2, Q))$
$\texttt{if}\ b\ \texttt{then}\ c_1\ \texttt{else}\ c_2$	$b \to \mathsf{wp}(c_1, Q) \wedge \neg b \to \mathsf{wp}(c_2, Q)$
$\texttt{while}\ b\ \texttt{do}\ c$	$I \wedge \forall \vec{y}. \Big((I \wedge b \to \mathsf{wp}(c, I)) \wedge (I \wedge \neg b \to Q) \Big) [\vec{x} \mapsto \vec{y}]$ ($\vec{x}$ are variables modified in c and I is the loop invariant)

Loop Invariant

$$\frac{\vdash \{A \wedge b\}\ c\ \{A\}}{\vdash\ \{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{A \wedge \neg b\}}$$

$$\mathsf{wp}(\texttt{while}\ b\ \texttt{do}\ c,\ Q) = I \wedge \forall \vec{y}. \Big((I \wedge b \rightarrow \mathsf{wp}(c, I)) \wedge (I \wedge \neg b \rightarrow Q) \Big) [\vec{x} \mapsto \vec{y}]$$

($\vec{x}$ are variables modified in c and I is the loop invariant)

- Unfortunate reality: Inferring invariants automatically is undecidable
- This puts significant limits on the degree to which we can automate verification

Loop Invariant

- Active research area:
  how to find loop invariants efficiently and automatically
- The simplest technique: guess-and-check!
- Given template of invariants (e.g., $? = ?$, $? \leq ?$), instantiate the holes with program variables and constants
- Check if it's an invariant; if not, try a different instantiation
- Abstract interpretation: popular approach to discover invariants
- Today we discuss an alternative approach: reducing verification to solving a set of Horn clauses

Motivating Example

```
x = 0;
while (x < N) {
  x = x + 2;
}
assert (x ≠ 13)
```

$$\frac{A \to I \qquad \vdash \{b \wedge I\}\ c\ \{I\} \qquad I \wedge \neg b \to B}{\vdash \{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{B\}}$$

To prove the Hoare triple $\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{B\}$ we need an invariant I

- I is *true* at the beginning: $A \to I$
- I is preserved by the loop: $\vdash \{b \wedge I\}\ c\ \{I\}$
- B is *true* after the loop: $I \wedge \neg b \to B$

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To prove the Hoare triple $\{A\}\ \texttt{while}\ b\ \texttt{do}\ c\ \{B\}$ we need an invariant $I(x)$

- I is *true* at the beginning: $A \to I$ $\qquad x = 0 \to I(x)$
- I is preserved by the loop: $\vdash \{b \wedge I\}\ c\ \{I\}$ $\qquad I(x) \wedge x < N \to I(x+2)$
- B is *true* after the loop: $I \wedge \neg b \to B$ $\qquad I(x) \wedge \neg(x < N) \to x \neq 13$

Motivating Example

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To prove the program we must **solve** for I:

$$I(x) \quad \Leftrightarrow \quad x \equiv 0 \pmod 2$$

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$$x = 0 \to I(x)$$
$$I(x) \wedge x < N \to I(x+2)$$
$$I(x) \wedge \neg(x < N) \to x \neq 13$$

"Horn Clause"
contains at most one positive literal

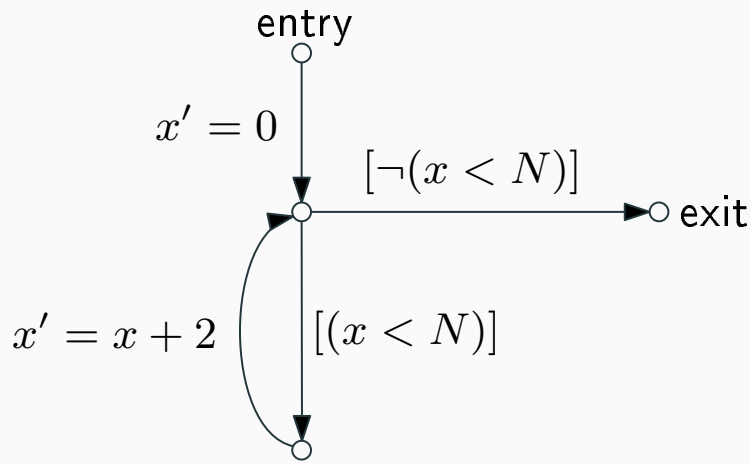
To prove the program we must **solve** for I:

$$I(x) \quad \Leftrightarrow \quad x \equiv 0 \pmod 2$$

Control Flow Graphs

- **Control Flow Graph (CFG):** graph representation of computation and control flow in the program
- Highlights the possible flow of execution
- Useful program representation for many software analysis tasks

```
x = 0;
while (x < N) {
   x = x + 2;
}
```



Control Flow Graphs

Control-Flow Graph (CFG) of a program P is a rooted directed graph $G = (V, E, \text{entry}, \text{exit})$ where

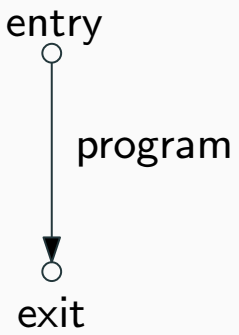
- $V \subseteq$ Label is a set of labels
- $E \subseteq$ Label $\times$ Action $\times$ Label is a set of arcs labeled by actions
- entry $\in V$ is the start state
- exit $\in V$ is the final state

Each action $f \in$ Action is a relation on program state:

$$[\![f]\!] \subseteq S \times S$$

Generating Control-Flow Graphs

- Start with graph that has one **entry** and one **exit** node
- Draw an edge from entry to exit and label it with the entire program

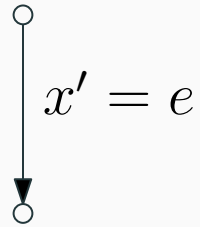


- Recursively decompose the program to have more edges with simpler labels
- When labels cannot be decomposed further, we are done

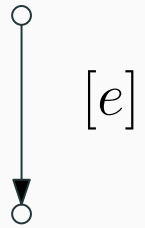
Basic Operations

- Base cases

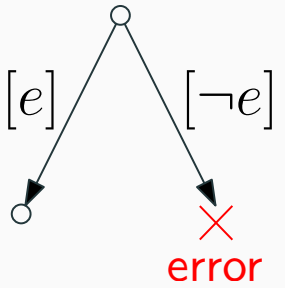
Assignment



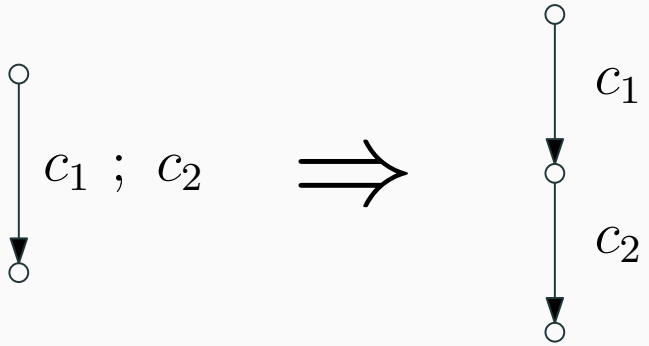
`assume`(e)



`assert`(e)

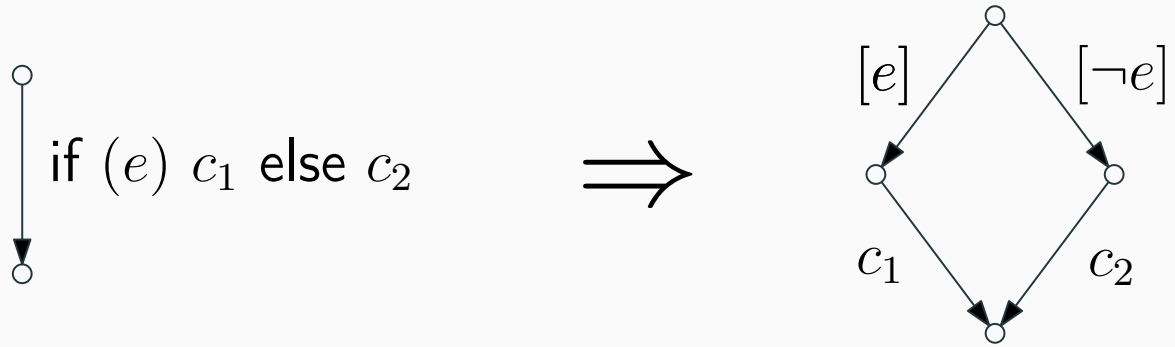


- Sequence of statements

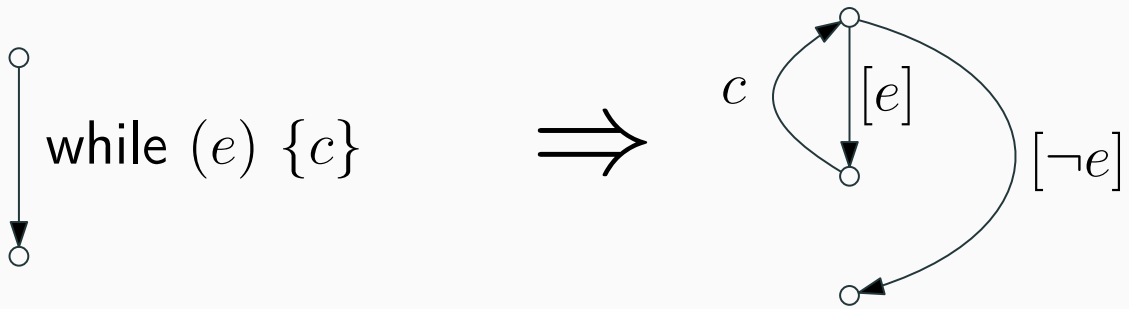


Control Structures

- Conditional statement



- While loop



Exercise: Convert to CFG

```
while(c2) {
  x = y - 1;
  y = z * 2;
  if (c1) x = y - z;
  z = 10;
}
z = x;
```

Exercise: Convert to CFG

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while(c2) {
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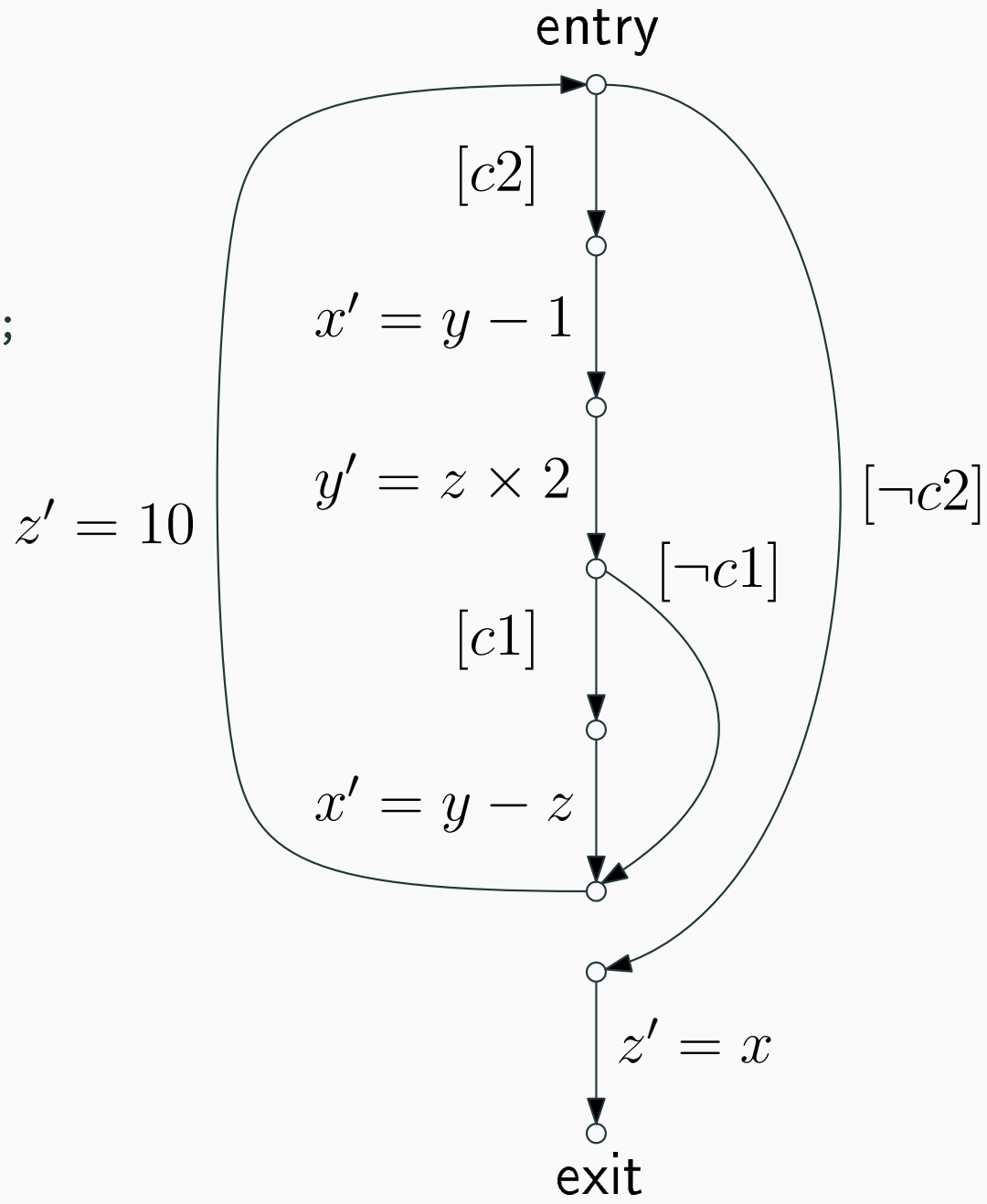

Example

- How to prove that the assertion does not fail in this program?

```
int n = 0;
while (true) {
  n = n + 1;
  assert(n≥−10);
  n = n − 1;
}
```

Example

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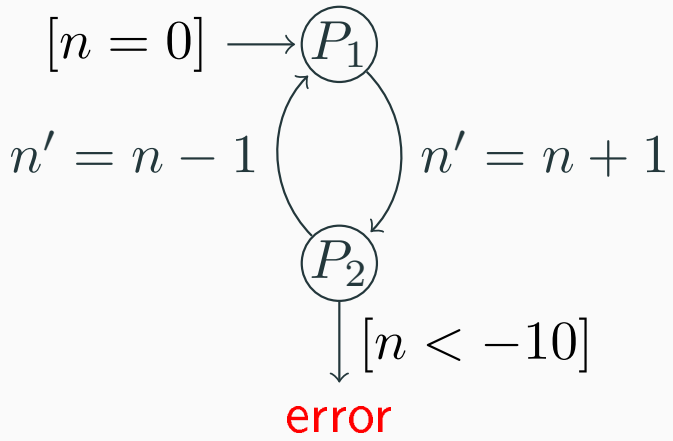
Control Flow Graph



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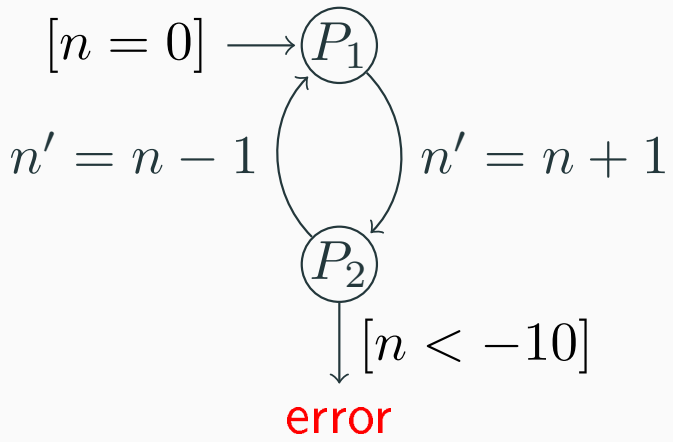

- Let $P_i(n)$ denotes a superset of reachable values of n in state P_i
- Initially we do not know the set of reachable values of n in each state
- P_i is a predicate on n, for example:
  - $P_1(n) = (n \geq 0)$ and $P_2(n) = (n = -5) \vee (n > 0)$
  - $P_1(n) = (n = 0)$ and $P_2(n) =$ true (any value can reach it)
  - ...
- We can write constraints between P_i's according to CFG

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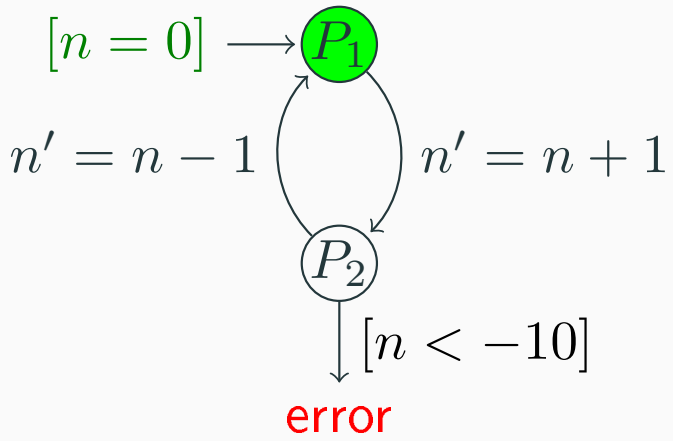
Control Flow Graph



$$(n = 0) \quad \rightarrow \quad P_1(n)$$

Example

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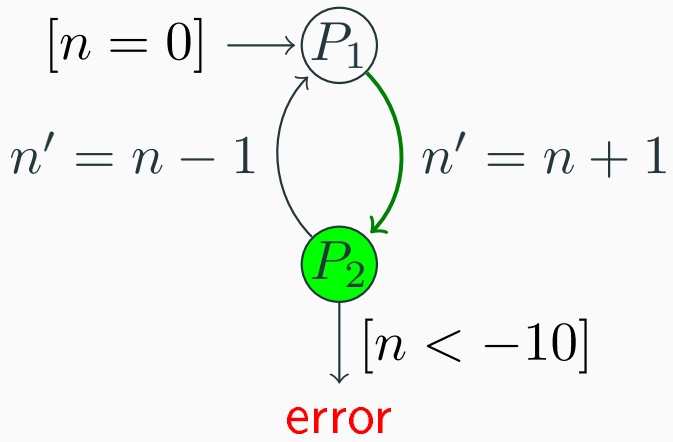
Control Flow Graph



$$(n = 0) \rightarrow P_1(n)$$
$$P_1(n) \wedge (n' = n + 1) \rightarrow P_2(n')$$

Example

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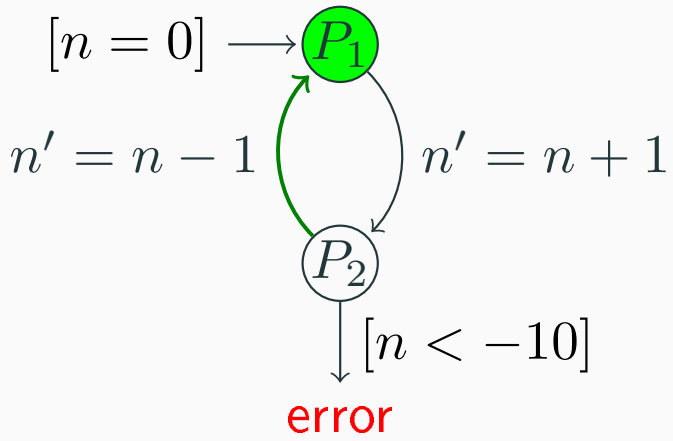
Control Flow Graph



$$
\begin{aligned}
(n = 0) &\rightarrow P_1(n) \\
P_1(n) \wedge (n' = n + 1) &\rightarrow P_2(n') \\
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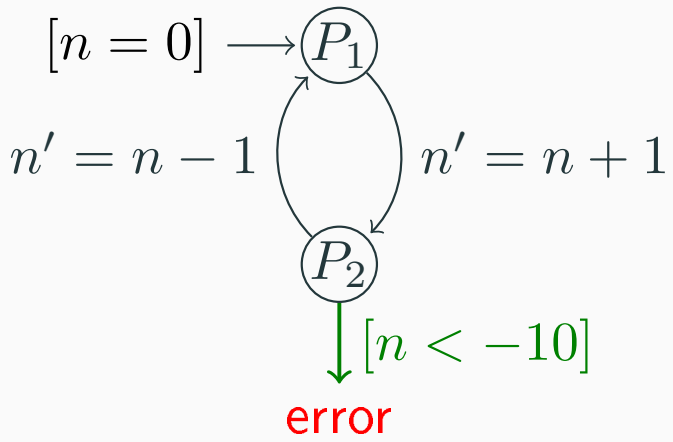

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\begin{aligned}
(n = 0) &\rightarrow P_1(n) \\
P_1(n) \wedge (n' = n + 1) &\rightarrow P_2(n') \\
P_2(n) \wedge (n' = n - 1) &\rightarrow P_1(n') \\
P_2(n) \wedge (n < -10) &\rightarrow \mathit{false}
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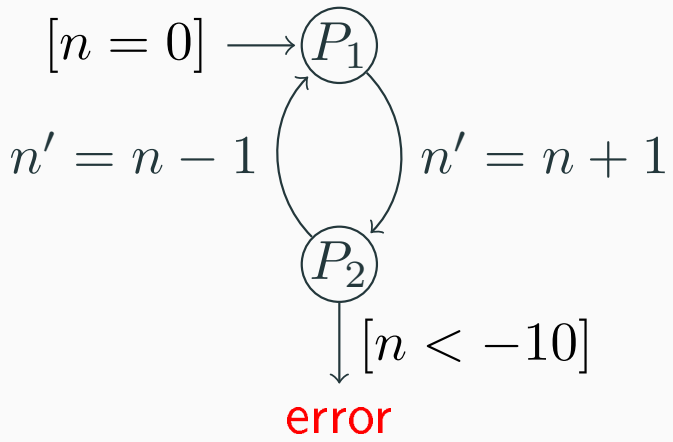

$$
\begin{array}{lll}
\forall n. & (n = 0) & \rightarrow \; P_1(n) \\
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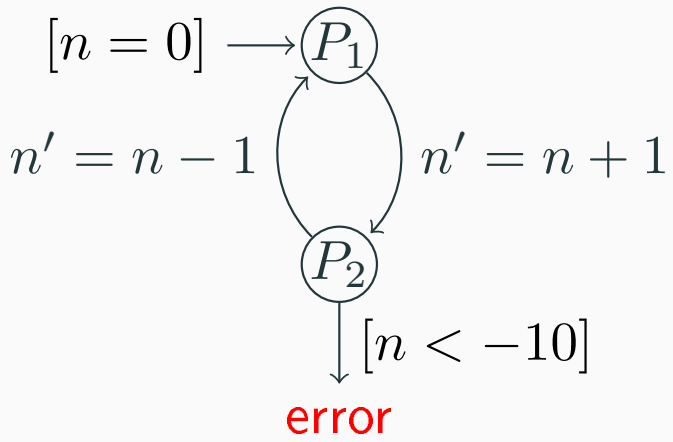

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\forall n. & (n = 0) & \rightarrow \; P_1(n) \\
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\forall n. & P_2(n) \wedge (n < -10) & \rightarrow \; \mathit{false}
\end{array}
$$

Solvable: $P_1(n) \equiv (n \geq 0)$ and $P_2(n) \equiv (n \geq 1)$

Demo

- Try `https://rise4fun.com/`

$$
\begin{array}{lll}
\forall n. & (n = 0) & \rightarrow P_1(n) \\
\forall n, n'. & P_1(n) \wedge (n' = n + 1) & \rightarrow P_2(n') \\
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\forall n. & P_2(n) \wedge (n < -10) & \rightarrow \mathit{false}
\end{array}
$$

```
(set-logic HORN)
(declare-fun P1 (Int) Bool)
(declare-fun P2 (Int) Bool)
(assert (forall ((n Int)) (=> (= n 0) (P1 n) )))
(assert (forall ((n Int)(np Int)) (=> (and (P1 n) (= np (+ n 1)))
                                      (P2 np))))
(assert (forall ((n Int)(np Int)) (=> (and (P2 n) (= np (- n 1)))
                                      (P1 np))))
(assert (forall ((n Int)) (=> (and (P2 n) (< n (- 10))) false)))
(check-sat)
(get-model)
```

Horn Clause

- Horn clause is an implication of the form:

$$\forall \bar{v}.\ \underbrace{\Phi(\bar{v}) \wedge R_1(\bar{v}) \wedge \cdots \wedge R_n(\bar{v})}_{\text{body}} \longrightarrow \underbrace{R_0(\bar{v})}_{\text{head}}$$

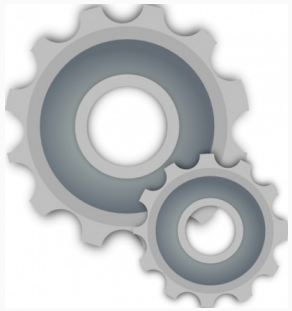
- $\Phi(\bar{v})$ is an arithmetic formula (e.g. $x + 2y \leq z$)
- $R_i(\bar{v})$ is a relation symbol
- Head of the clause is either a relation symbol or false
- A solution for the Horn clause is an assignment of formulae to relation symbols for which the implication is valid

Verification by Solving Horn Clauses

Code

Safety Description

$$\forall \bar{v}.\ \Phi^0(\bar{v}) \wedge R_1^0(\bar{v}) \wedge \cdots \wedge R_n^0(\bar{v}) \rightarrow R_0^0(\bar{v})$$
$$\forall \bar{v}.\ \Phi^1(\bar{v}) \wedge R_1^1(\bar{v}) \wedge \cdots \wedge R_n^1(\bar{v}) \rightarrow R_0^1(\bar{v})$$
$$\vdots$$
$$\forall \bar{v}.\ \Phi^m(\bar{v}) \wedge R_1^m(\bar{v}) \wedge \cdots \wedge R_n^m(\bar{v}) \rightarrow R_0^m(\bar{v})$$
$$\forall \bar{v}.\ \Phi^i(\bar{v}) \wedge R_1^i(\bar{v}) \wedge \cdots \wedge R_n^i(\bar{v}) \rightarrow \mathit{false}$$



Horn Clause Solver

Exercise

- Convert to Horn clauses

```
int x,y;
assume(x≥0 ∧ y≥0);
while(x≠y) {
 if (x>y) then x:=x−y;
 else y:=y−x;
}
assert(x≠−1);
```

Exercise

- Convert to Horn clauses

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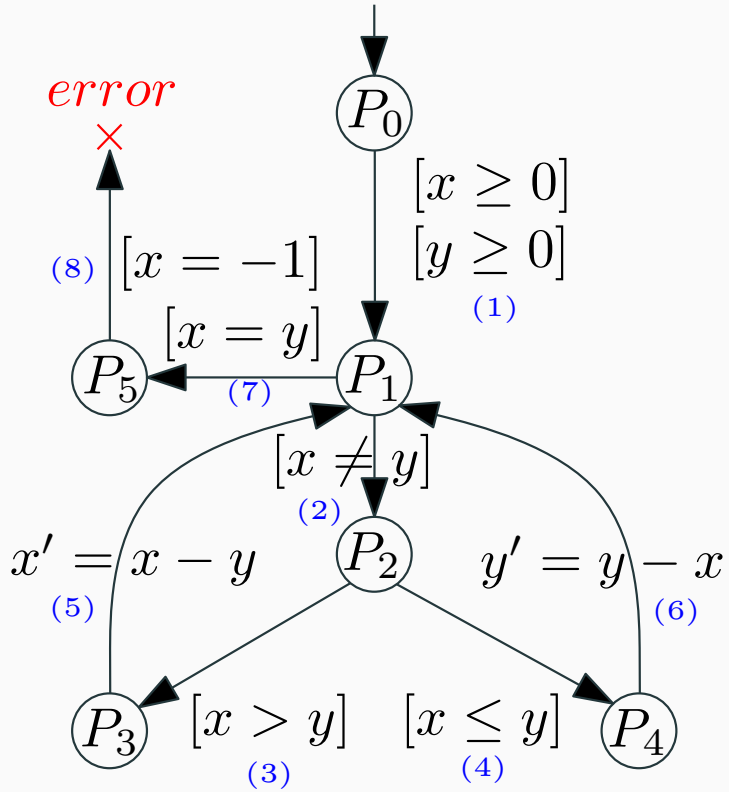

Exercise

- Convert to Horn clauses

$$
\begin{array}{lcc}
 & \mathit{true} & \rightarrow & P_0(x,y) \\
1) & P_0(x,y) \wedge (x \geq 0) \wedge (y \geq 0) & \rightarrow & P_1(x,y) \\
2) & P_1(x,y) \wedge (x \neq y) & \rightarrow & P_2(x,y) \\
3) & P_2(x,y) \wedge (x > y) & \rightarrow & P_3(x,y) \\
4) & P_2(x,y) \wedge (x \leq y) & \rightarrow & P_4(x,y) \\
5) & P_3(x,y) \wedge (x' = x - y) & \rightarrow & P_1(x',y) \\
6) & P_4(x,y) \wedge (y' = y - x) & \rightarrow & P_1(x,y') \\
7) & P_1(x,y) \wedge (x = y) & \rightarrow & P_5(x,y) \\
8) & P_5(x,y) \wedge (x = -1) & \rightarrow & \mathit{false}
\end{array}
$$